

INVARIANT DIFFERENTIAL OPERATORS ON SIEGEL-JACOBI SPACE

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ABSTRACT. For two positive integers m and n , we let \mathbb{H}_n be the Siegel upper half plane of degree n and let $\mathbb{C}^{(m,n)}$ be the set of all $m \times n$ complex matrices. In this article, we study differential operators on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ that are invariant under the *natural* action of the Jacobi group $Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$ on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$, where $H_{\mathbb{R}}^{(n,m)}$ denotes the Heisenberg group. We give some explicit invariant differential operators. We present important problems which are natural. We give some partial solutions for these natural problems.

1. Introduction

For a given fixed positive integer n , we let

$$\mathbb{H}_n = \{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \text{Im } \Omega > 0 \}$$

be the Siegel upper half plane of degree n and let

$$Sp(n, \mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n \}$$

be the symplectic group of degree n , where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l , tM denotes the transpose matrix of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n, \mathbb{R})$ acts on \mathbb{H}_n transitively by

$$(1.1) \quad M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers m and n , we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu {}^t\lambda \text{ symmetric} \}$$

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endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda {}^t\mu' - \mu {}^t\lambda')$$

with $(\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$. We define the semidirect product of $Sp(n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J = Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda} {}^t\mu' - \tilde{\mu} {}^t\lambda'))$$

with $M, M' \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$(1.2) \quad (M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left(M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} \right),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and that the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is not a symmetric space. We refer to [1, 6, 22, 23, 24, 25, 27, 28, 29, 30, 31] about automorphic forms on G^J and topics related to the content of this paper. From now on, for brevity we write $\mathbb{H}_{n,m} = \mathbb{H}_n \times \mathbb{C}^{(m,n)}$, called the Siegel-Jacobi space of degree n and index m .

The aim of this paper is to study differential operators on $\mathbb{H}_{n,m}$ which are invariant under the *natural* action (1.2) of G^J . The study of these invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$ is interesting and important in the aspects of invariant theory, arithmetic and geometry. This article is organized as follows. In Section 2, we review differential operators on \mathbb{H}_n invariant under the action (1.1) of $Sp(n, \mathbb{R})$. We let $\mathbb{D}(\mathbb{H}_n)$ denote the algebra of all differential operators on \mathbb{H}_n that are invariant under the action (1.1). According to the work of Harish-Chandra [7, 8], we see that $\mathbb{D}(\mathbb{H}_n)$ is a commutative algebra which is isomorphic to the center of the universal enveloping algebra of the complexification of the Lie algebra of $Sp(n, \mathbb{R})$. We briefly describe the work of Maass [14] about constructing explicit algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$ and Shimura's construction [18] of canonically defined algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$. In Section 3, we study differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J . For two positive integers m and n , we let

$$T_{n,m} = \{ (\omega, z) \mid \omega = {}^t\omega \in \mathbb{C}^{(n,n)}, z \in \mathbb{C}^{(m,n)} \}$$

be the complex vector space of dimension $\frac{n(n+1)}{2} + mn$. From the adjoint action of the Jacobi group G^J , we have the *natural action* of the unitary group $U(n)$ on $T_{n,m}$ given by

$$(1.3) \quad u \cdot (\omega, z) = (u \omega {}^tu, {}^tz u), \quad u \in U(n), (\omega, z) \in T_{n,m}.$$

The action (1.3) of $U(n)$ induces canonically the representation τ of $U(n)$ on the polynomial algebra $\text{Pol}(T_{n,m})$ consisting of complex valued polynomial functions on $T_{n,m}$. Let $\text{Pol}(T_{n,m})^{U(n)}$ denote the subalgebra of $\text{Pol}(T_{n,m})$ consisting of all polynomials on $T_{n,m}$ invariant under the representation τ of $U(n)$, and $\mathbb{D}(\mathbb{H}_{n,m})$ denote the algebra of all differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J . We see that there is a canonically defined linear bijection of $\text{Pol}(T_{n,m})^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$ which is not multiplicative. We will see that $\mathbb{D}(\mathbb{H}_{n,m})$ is *not* commutative. The main important problem is to find explicit generators of $\text{Pol}(T_{n,m})^{U(n)}$ and explicit generators of $\mathbb{D}(\mathbb{H}_{n,m})$. We propose several natural problems. We want to mention that at this moment it is quite complicated and difficult to find the explicit generators of $\mathbb{D}(\mathbb{H}_{n,m})$ and to express invariant differential operators on $\mathbb{H}_{n,m}$ explicitly. In Section 4, we give some examples of explicit G^J -invariant differential operators on $\mathbb{H}_{n,m}$ that are obtained by complicated calculations. In Section 5, we deal with the special case $n = m = 1$ in detail. We give complete solutions of the problems that are proposed in Section 3. In Section 6, we deal with the case that $n = 1$ and m is arbitrary. We give some partial solutions for the problems proposed in Section 3. In the final section, using these invariant differential operators on the Siegel-Jacobi space, we discuss a notion of Maass-Jacobi forms.

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Notations: We denote by \mathbb{Q} , \mathbb{R} and \mathbb{C} the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by \mathbb{Z} and \mathbb{Z}^+ the ring of integers and the set of all positive integers respectively. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers k and l , $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F . For a square matrix $A \in F^{(k,k)}$ of degree k , $\text{tr}(A)$ denotes the trace of A . For any $M \in F^{(k,l)}$, tM denotes the transpose matrix of M . I_n denotes the identity matrix of degree n . For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A , \overline{A} denotes the complex *conjugate* of A . For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a positive integer n , I_n denotes the identity matrix of degree n .

2. Invariant Differential Operators on the Siegel Space

For a coordinate $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\bar{\Omega} = (d\bar{\omega}_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{\Omega}} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \bar{\omega}_{ij}} \right).$$

Then for a positive real number A ,

$$(2.1) \quad ds_{n;A}^2 = A \operatorname{tr} \left(Y^{-1} d\Omega Y^{-1} d\bar{\Omega} \right)$$

is a $Sp(n, \mathbb{R})$ -invariant Kähler metric on \mathbb{H}_n (cf. [19, 20]), where $\operatorname{tr}(M)$ denotes the trace of a square matrix M . H. Maass [13] proved that the Laplacian of $ds_{n;A}^2$ is given by

$$(2.2) \quad \Delta_{n;A} = \frac{4}{A} \operatorname{tr} \left(Y^t \left(Y \frac{\partial}{\partial \bar{\Omega}} \right) \frac{\partial}{\partial \Omega} \right).$$

And

$$dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1 \leq i \leq j \leq n} dx_{ij} \prod_{1 \leq i \leq j \leq n} dy_{ij}$$

is a $Sp(n, \mathbb{R})$ -invariant volume element on \mathbb{H}_n (cf. [20, p. 130]).

For brevity, we write $G = Sp(n, \mathbb{R})$. The isotropy subgroup K at iI_n for the action (1.1) is a maximal compact subgroup given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t A + B^t B = I_n, \ A^t B = B^t A, \ A, B \in \mathbb{R}^{(n,n)} \right\}.$$

Let \mathfrak{k} be the Lie algebra of K . Then the Lie algebra \mathfrak{g} of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^t X_1 \end{pmatrix} \mid X_1, X_2, X_3 \in \mathbb{R}^{(n,n)}, \ X_2 = {}^t X_2, \ X_3 = {}^t X_3 \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \mid {}^t X + X = 0, \ Y = {}^t Y \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \mid X = {}^t X, \ Y = {}^t Y, \ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace \mathfrak{p} of \mathfrak{g} may be regarded as the tangent space of \mathbb{H}_n at iI_n . The adjoint representation of G on \mathfrak{g} induces the action of K on \mathfrak{p} given by

$$(2.3) \quad k \cdot Z = k Z {}^t k, \quad k \in K, \ Z \in \mathfrak{p}.$$

Let T_n be the vector space of $n \times n$ symmetric complex matrices. We let $\Psi : \mathfrak{p} \longrightarrow T_n$ be the map defined by

$$(2.4) \quad \Psi \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \right) = X + iY, \quad \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}.$$

We let $\delta : K \longrightarrow U(n)$ be the isomorphism defined by

$$(2.5) \quad \delta \left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right) = A + iB, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K,$$

where $U(n)$ denotes the unitary group of degree n . We identify \mathfrak{p} (resp. K) with T_n (resp. $U(n)$) through the map Ψ (resp. δ). We consider the action of $U(n)$ on T_n defined by

$$(2.6) \quad h \cdot \omega = h\omega^t h, \quad h \in U(n), \omega \in T_n.$$

Then the adjoint action (2.3) of K on \mathfrak{p} is compatible with the action (2.6) of $U(n)$ on T_n through the map Ψ . Precisely for any $k \in K$ and $Z \in \mathfrak{p}$, we get

$$(2.7) \quad \Psi(k Z^t k) = \delta(k) \Psi(Z)^t \delta(k).$$

The action (2.6) induces the action of $U(n)$ on the polynomial algebra $\text{Pol}(T_n)$ and the symmetric algebra $S(T_n)$ respectively. We denote by $\text{Pol}(T_n)^{U(n)}$ (resp. $S(T_n)^{U(n)}$) the subalgebra of $\text{Pol}(T_n)$ (resp. $S(T_n)$) consisting of $U(n)$ -invariants. The following inner product $(\ , \)$ on T_n defined by

$$(Z, W) = \text{tr}(Z \overline{W}), \quad Z, W \in T_n$$

gives an isomorphism as vector spaces

$$(2.8) \quad T_n \cong T_n^*, \quad Z \mapsto f_Z, \quad Z \in T_n,$$

where T_n^* denotes the dual space of T_n and f_Z is the linear functional on T_n defined by

$$f_Z(W) = (W, Z), \quad W \in T_n.$$

It is known that there is a canonical linear bijection of $S(T_n)^{U(n)}$ onto the algebra $\mathbb{D}(\mathbb{H}_n)$ of differential operators on \mathbb{H}_n invariant under the action (1.1) of G . Identifying T_n with T_n^* by the above isomorphism (2.8), we get a canonical linear bijection

$$(2.9) \quad \Theta_n : \text{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_n)$$

of $\text{Pol}(T_n)^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_n)$. The map Θ_n is described explicitly as follows. Similarly the action (2.3) induces the action of K on the polynomial algebra $\text{Pol}(\mathfrak{p})$ and the symmetric algebra $S(\mathfrak{p})$ respectively. Through the map Ψ , the subalgebra $\text{Pol}(\mathfrak{p})^K$ of $\text{Pol}(\mathfrak{p})$ consisting of K -invariants is isomorphic to $\text{Pol}(T_n)^{U(n)}$. We put $N = n(n+1)$. Let $\{\xi_\alpha \mid 1 \leq \alpha \leq N\}$ be a basis of a real vector space \mathfrak{p} . If $P \in \text{Pol}(\mathfrak{p})^K$, then

$$(2.10) \quad (\Theta_n(P)f)(gK) = \left[P \left(\frac{\partial}{\partial t_\alpha} \right) f \left(g \exp \left(\sum_{\alpha=1}^N t_\alpha \xi_\alpha \right) K \right) \right]_{(t_\alpha)=0},$$

where $f \in C^\infty(\mathbb{H}_n)$. We refer to [9, 10] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P \in \text{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [7, 8], the algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators and is isomorphic to the commutative ring

$\mathbb{C}[x_1, \dots, x_n]$ with n indeterminates. We note that n is the real rank of G . Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of \mathfrak{g} . It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Using a classical invariant theory (cf. [11, 21]), we can show that $\text{Pol}(T_n)^{U(n)}$ is generated by the following algebraically independent polynomials

$$(2.11) \quad q_j(\omega) = \text{tr}\left((\omega \bar{\omega})^j\right), \quad \omega \in T_n, \quad j = 1, 2, \dots, n.$$

For each j with $1 \leq j \leq n$, the image $\Theta_n(q_j)$ of q_j is an invariant differential operator on \mathbb{H}_n of degree $2j$. The algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators $\Theta_n(q_1), \Theta_n(q_2), \dots, \Theta_n(q_n)$. In particular,

$$(2.12) \quad \Theta_n(q_1) = c_1 \text{tr}\left(Y^t \left(Y \frac{\partial}{\partial \bar{\Omega}}\right) \frac{\partial}{\partial \Omega}\right) \quad \text{for some constant } c_1.$$

We observe that if we take $\omega = x + iy \in T_n$ with real x, y , then $q_1(\omega) = q_1(x, y) = \text{tr}(x^2 + y^2)$ and

$$q_2(\omega) = q_2(x, y) = \text{tr}\left((x^2 + y^2)^2 + 2x(xy - yx)y\right).$$

It is a natural question to express the images $\Theta_n(q_j)$ explicitly for $j = 2, 3, \dots, n$. We hope that the images $\Theta_n(q_j)$ for $j = 2, 3, \dots, n$ are expressed in the form of the *trace* as $\Phi(q_1)$.

H. Maass [14] found algebraically independent generators H_1, H_2, \dots, H_n of $\mathbb{D}(\mathbb{H}_n)$. We will describe H_1, H_2, \dots, H_n explicitly. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega = X + iY \in \mathbb{H}_n$ with real X, Y , we set

$$\Omega_* = M \cdot \Omega = X_* + iY_* \quad \text{with } X_*, Y_* \text{ real.}$$

We set

$$\begin{aligned} K &= (\Omega - \bar{\Omega}) \frac{\partial}{\partial \Omega} = 2iY \frac{\partial}{\partial \Omega}, \\ \Lambda &= (\Omega - \bar{\Omega}) \frac{\partial}{\partial \bar{\Omega}} = 2iY \frac{\partial}{\partial \bar{\Omega}}, \\ K_* &= (\Omega_* - \bar{\Omega}_*) \frac{\partial}{\partial \Omega_*} = 2iY_* \frac{\partial}{\partial \Omega_*}, \\ \Lambda_* &= (\Omega_* - \bar{\Omega}_*) \frac{\partial}{\partial \bar{\Omega}_*} = 2iY_* \frac{\partial}{\partial \bar{\Omega}_*}. \end{aligned}$$

Then it is easily seen that

$$(2.13) \quad K_* = {}^t(C\bar{\Omega} + D)^{-1} {}^t\{(C\Omega + D) {}^tK\},$$

$$(2.14) \quad \Lambda_* = {}^t(C\Omega + D)^{-1} {}^t\{(C\bar{\Omega} + D) {}^t\Lambda\}$$

and

$$(2.15) \quad {}^t\{(C\overline{\Omega} + D) {}^t\Lambda\} = \Lambda {}^t(C\overline{\Omega} + D) - \frac{n+1}{2} (\Omega - \overline{\Omega}) {}^tC.$$

Using Formulas (2.13), (2.14) and (2.15), we can show that

$$(2.16) \quad \Lambda_* K_* + \frac{n+1}{2} K_* = {}^t(C\Omega + D)^{-1} \left\{ (C\Omega + D) \left(\Lambda K + \frac{n+1}{2} K \right) \right\}.$$

Therefore we get

$$(2.17) \quad \text{tr} \left(\Lambda_* K_* + \frac{n+1}{2} K_* \right) = \text{tr} \left(\Lambda K + \frac{n+1}{2} K \right).$$

We set

$$(2.18) \quad A^{(1)} = \Lambda K + \frac{n+1}{2} K.$$

We define $A^{(j)}$ ($j = 2, 3, \dots, n$) recursively by

$$(2.19) \quad \begin{aligned} A^{(j)} &= A^{(1)} A^{(j-1)} - \frac{n+1}{2} \Lambda A^{(j-1)} + \frac{1}{2} \Lambda \text{tr}(A^{(j-1)}) \\ &\quad + \frac{1}{2} (\Omega - \overline{\Omega}) \left\{ (\Omega - \overline{\Omega})^{-1} {}^t({}^t\Lambda A^{(j-1)}) \right\}. \end{aligned}$$

We set

$$(2.20) \quad H_j = \text{tr}(A^{(j)}), \quad j = 1, 2, \dots, n.$$

As mentioned before, Maass proved that H_1, H_2, \dots, H_n are algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$.

In fact, we see that

$$(2.21) \quad -H_1 = \Delta_{n;1} = 4 \text{tr} \left(Y {}^t \left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right).$$

is the Laplacian for the invariant metric $ds_{n;1}^2$ on \mathbb{H}_n .

Conjecture. For $j = 2, 3, \dots, n$, $\Theta_n(q_j) = c_j H_j$ for a suitable constant c_j .

Example 2.1. We consider the case $n = 1$. The algebra $\text{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$q(\omega) = \omega \overline{\omega}, \quad \omega = x + iy \in \mathbb{C} \text{ with } x, y \text{ real.}$$

Using Formula (2.10), we get

$$\Theta_1(q) = 4y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}[\Theta_1(q)] = \mathbb{C}[H_1]$.

Example 2.2. We consider the case $n = 2$. The algebra $\text{Pol}(T_2)^{U(2)}$ is generated by the polynomial

$$q_1(\omega) = \text{tr}(\omega \bar{\omega}), \quad q_2(\omega) = \text{tr}\left((\omega \bar{\omega})^2\right), \quad \omega \in T_2.$$

Using Formula (2.10), we may express $\Theta_2(q_1)$ and $\Theta_2(q_2)$ explicitly. $\Theta_2(q_1)$ is expressed by Formula (2.12). The computation of $\Theta_2(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Theta_2(q_2)$ was essentially computed in [4], Proposition 6. Therefore

$$\mathbb{D}(\mathbb{H}_2) = \mathbb{C}[\Theta_2(q_1), \Theta_2(q_2)] = \mathbb{C}[H_1, H_2].$$

In fact, the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ was computed in [4].

G. Shimura [18] found canonically defined algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$. We will describe his way of constructing those generators roughly. Let $K_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}, \dots$ denote the complexification of $K, \mathfrak{g}, \mathfrak{k}, \mathfrak{p}, \dots$ respectively. Then we have the Cartan decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}, \quad \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^+ + \mathfrak{p}_{\mathbb{C}}^-$$

with the properties

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}^{\pm}] \subset \mathfrak{p}_{\mathbb{C}}^{\pm}, \quad [\mathfrak{p}_{\mathbb{C}}^+, \mathfrak{p}_{\mathbb{C}}^+] = [\mathfrak{p}_{\mathbb{C}}^-, \mathfrak{p}_{\mathbb{C}}^-] = \{0\}, \quad [\mathfrak{p}_{\mathbb{C}}^+, \mathfrak{p}_{\mathbb{C}}^-] = \mathfrak{k}_{\mathbb{C}},$$

where

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &= \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^t X_1 \end{pmatrix} \mid X_1, X_2, X_3 \in \mathbb{C}^{(n,n)}, X_2 = {}^t X_2, X_3 = {}^t X_3 \right\}, \\ \mathfrak{k}_{\mathbb{C}} &= \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \mid {}^t A + A = 0, B = {}^t B \right\}, \\ \mathfrak{p}_{\mathbb{C}} &= \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \mid X = {}^t X, Y = {}^t Y \right\}, \\ \mathfrak{p}_{\mathbb{C}}^+ &= \left\{ \begin{pmatrix} Z & iZ \\ iZ & -Z \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \mid Z = {}^t Z \in \mathbb{C}^{(n,n)} \right\}, \\ \mathfrak{p}_{\mathbb{C}}^- &= \left\{ \begin{pmatrix} Z & -iZ \\ -iZ & -Z \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \mid Z = {}^t Z \in \mathbb{C}^{(n,n)} \right\}. \end{aligned}$$

For a complex vector space W and a nonnegative integer r , we denote by $\text{Pol}_r(W)$ the vector space of complex-valued homogeneous polynomial functions on W of degree r . We put

$$\text{Pol}^r(W) := \sum_{s=0}^r \text{Pol}_s(W).$$

$\text{ML}_r(W)$ denotes the vector space of all \mathbb{C} -multilinear maps of $W \times \dots \times W$ (r copies) into \mathbb{C} . An element Q of $\text{ML}_r(W)$ is called *symmetric* if

$$Q(x_1, \dots, x_r) = Q(x_{\pi(1)}, \dots, x_{\pi(r)})$$

for each permutation π of $\{1, 2, \dots, r\}$. Given $P \in \text{Pol}_r(W)$, there is a unique element symmetric element P_* of $\text{Ml}_r(W)$ such that

$$(2.22) \quad P(x) = P_*(x, \dots, x) \quad \text{for all } x \in W.$$

Moreover the map $P \mapsto P_*$ is a \mathbb{C} -linear bijection of $\text{Pol}_r(W)$ onto the set of all symmetric elements of $\text{Ml}_r(W)$. We let $S_r(W)$ denote the subspace consisting of all homogeneous elements of degree r in the symmetric algebra $S(W)$. We note that $\text{Pol}_r(W)$ and $S_r(W)$ are dual to each other with respect to the pairing

$$(2.23) \quad \langle \alpha, x_1 \cdots x_r \rangle = \alpha_*(x_1, \dots, x_r) \quad (x_i \in W, \alpha \in \text{Pol}_r(W)).$$

Let $\mathfrak{p}_{\mathbb{C}}^*$ be the dual space of $\mathfrak{p}_{\mathbb{C}}$, that is, $\mathfrak{p}_{\mathbb{C}}^* = \text{Pol}_1(\mathfrak{p}_{\mathbb{C}})$. Let $\{X_1, \dots, X_N\}$ be a basis of $\mathfrak{p}_{\mathbb{C}}$ and $\{Y_1, \dots, Y_N\}$ be the basis of $\mathfrak{p}_{\mathbb{C}}^*$ dual to $\{X_\nu\}$, where $N = n(n+1)$. We note that $\text{Pol}_r(\mathfrak{p}_{\mathbb{C}})$ and $\text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ are dual to each other with respect to the pairing

$$(2.24) \quad \langle \alpha, \beta \rangle = \sum \alpha_*(X_{i_1}, \dots, X_{i_r}) \beta_*(Y_{i_1}, \dots, Y_{i_r}),$$

where $\alpha \in \text{Pol}_r(\mathfrak{p}_{\mathbb{C}})$, $\beta \in \text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ and (i_1, \dots, i_r) runs over $\{1, \dots, N\}^r$. Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathcal{U}^p(\mathfrak{g}_{\mathbb{C}})$ its subspace spanned by the elements of the form $V_1 \cdots V_s$ with $V_i \in \mathfrak{g}_{\mathbb{C}}$ and $s \leq p$. We recall that there is a \mathbb{C} -linear bijection ψ of the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ onto $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ which is characterized by the property that $\psi(X^r) = X^r$ for all $X \in \mathfrak{g}_{\mathbb{C}}$. For each $\alpha \in \text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ we define an element $\omega(\alpha)$ of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ by

$$(2.25) \quad \omega(\alpha) := \sum \alpha_*(Y_{i_1}, \dots, Y_{i_r}) X_{i_1} \cdots X_{i_r},$$

where (i_1, \dots, i_r) runs over $\{1, \dots, N\}^r$. If $Y \in \mathfrak{p}_{\mathbb{C}}$, then Y^r as an element of $\text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ is defined by

$$Y^r(u) = Y(u)^r \quad \text{for all } u \in \mathfrak{p}_{\mathbb{C}}^*.$$

Hence $(Y^r)_*(u_1, \dots, u_r) = Y(u_1) \cdots Y(u_r)$. According to (2.25), we see that if $\alpha(\sum t_i Y_i) = P(t_1, \dots, t_N)$ for $t_i \in \mathbb{C}$ with a polynomial P , then

$$(2.26) \quad \omega(\alpha) = \psi(P(X_1, \dots, X_N)).$$

Thus ω is a \mathbb{C} -linear injection of $\text{Pol}(\mathfrak{p}_{\mathbb{C}}^*)$ into $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ independent of the choice of a basis. We observe that $\omega(\text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)) = \psi(S_r(\mathfrak{p}_{\mathbb{C}}))$. It is a well-known fact that if $\alpha_1, \dots, \alpha_m \in \text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$, then

$$(2.27) \quad \omega(\alpha_1 \cdots \alpha_m) - \omega(\alpha_m) \cdots \omega(\alpha_1) \in \mathcal{U}^{r-1}(\mathfrak{g}_{\mathbb{C}}).$$

We have a canonical pairing

$$(2.28) \quad \langle \cdot, \cdot \rangle : \text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^+) \times \text{Pol}_r(\mathfrak{p}_{\mathbb{C}}^-) \longrightarrow \mathbb{C}$$

defined by

$$(2.29) \quad \langle f, g \rangle = \sum f_*(\tilde{X}_{i_1}, \dots, \tilde{X}_{i_r}) g_*(\tilde{Y}_{i_1}, \dots, \tilde{Y}_{i_r}),$$

where f_* (resp. g_*) are the unique symmetric elements of $\text{Ml}_r(\mathfrak{p}_{\mathbb{C}}^+)$ (resp. $\text{Ml}_r(\mathfrak{p}_{\mathbb{C}}^-)$), and $\{\tilde{X}_1, \dots, \tilde{X}_{\tilde{N}}\}$ and $\{\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{N}}\}$ are dual bases of $\mathfrak{p}_{\mathbb{C}}^+$ and $\mathfrak{p}_{\mathbb{C}}^-$ with respect to

the Killing form $B(X, Y) = 2(n+1) \operatorname{tr}(XY)$, $\tilde{N} = \frac{n(n+1)}{2}$, and (i_1, \dots, i_r) runs over $\{1, \dots, \tilde{N}\}^r$.

The adjoint representation of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}^{\pm}$ induces the representation of $K_{\mathbb{C}}$ on $\operatorname{Pol}_r(\mathfrak{p}_{\mathbb{C}}^{\pm})$. Given a $K_{\mathbb{C}}$ -irreducible subspace Z of $\operatorname{Pol}_r(\mathfrak{p}_{\mathbb{C}}^+)$, we can find a unique $K_{\mathbb{C}}$ -irreducible subspace W of $\operatorname{Pol}_r(\mathfrak{p}_{\mathbb{C}}^-)$ such that $\operatorname{Pol}_r(\mathfrak{p}_{\mathbb{C}}^-)$ is the direct sum of W and the annihilator of Z . Then Z and W are dual with respect to the pairing (2.28). Take bases $\{\zeta_1, \dots, \zeta_{\kappa}\}$ of Z and $\{\xi_1, \dots, \xi_{\kappa}\}$ of W that are dual to each other. We set

$$(2.30) \quad f_Z(x, y) = \sum_{\nu=1}^{\kappa} \zeta_{\nu}(x) \xi_{\nu}(y) \quad (x \in \mathfrak{p}_{\mathbb{C}}^+, y \in \mathfrak{p}_{\mathbb{C}}^-).$$

It is easily seen that f_Z belongs to $\operatorname{Pol}_{2r}(\mathfrak{p}_{\mathbb{C}})^K$ and is independent of the choice of dual bases $\{\zeta_{\nu}\}$ and $\{\xi_{\nu}\}$. Shimura [18] proved that there exists a canonically defined set $\{Z_1, \dots, Z_n\}$ with a $K_{\mathbb{C}}$ -irreducible subspace Z_r of $\operatorname{Pol}_r(\mathfrak{p}_{\mathbb{C}}^+)$ ($1 \leq r \leq n$) such that f_{Z_1}, \dots, f_{Z_n} are algebraically independent generators of $\operatorname{Pol}(\mathfrak{p}_{\mathbb{C}})^K$. We can identify $\mathfrak{p}_{\mathbb{C}}^+$ with T_n . We recall that T_n denotes the vector space of $n \times n$ symmetric complex matrices. We can take Z_r as the subspace of $\operatorname{Pol}_r(T_n)$ spanned by the functions $f_{a,r}(Z) = \det_r({}^t a Z a)$ for all $a \in GL(n, \mathbb{C})$, where $\det_r(x)$ denotes the determinant of the upper left $r \times r$ submatrix of x . For every $f \in \operatorname{Pol}(\mathfrak{p}_{\mathbb{C}})^K$, we let $\Omega(f)$ denote the element of $\mathbb{D}(\mathbb{H}_n)$ represented by $\omega(f)$. Then $\mathbb{D}(\mathbb{H}_n)$ is the polynomial ring $\mathbb{C}[\omega(f_{Z_1}), \dots, \omega(f_{Z_n})]$ generated by n algebraically independent elements $\omega(f_{Z_1}), \dots, \omega(f_{Z_n})$.

3. Invariant Differential Operators on Siegel-Jacobi Space

The stabilizer K^J of G^J at $(iI_n, 0)$ is given by

$$K^J = \left\{ (k, (0, 0; \kappa)) \mid k \in K, \kappa = {}^t \kappa \in \mathbb{R}^{(m, m)} \right\}.$$

Therefore $\mathbb{H}_{n,m} \cong G^J/K^J$ is a homogeneous space of *non-reductive type*. The Lie algebra \mathfrak{g}^J of G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\mathfrak{g}^J = \left\{ (Z, (P, Q, R)) \mid Z \in \mathfrak{g}, P, Q \in \mathbb{R}^{(m, n)}, R = {}^t R \in \mathbb{R}^{(m, m)} \right\},$$

$$\mathfrak{k}^J = \left\{ (X, (0, 0, R)) \mid X \in \mathfrak{k}, R = {}^t R \in \mathbb{R}^{(m, m)} \right\},$$

$$\mathfrak{p}^J = \left\{ (Y, (P, Q, 0)) \mid Y \in \mathfrak{p}, P, Q \in \mathbb{R}^{(m, n)} \right\}.$$

Thus the tangent space of the homogeneous space $\mathbb{H}_{n,m}$ at $(iI_n, 0)$ is identified with \mathfrak{p}^J .

If $\alpha = \left(\begin{pmatrix} X_1 & Y_1 \\ Z_1 & -X_1 \end{pmatrix}, (P_1, Q_1, R_1) \right)$ and $\beta = \left(\begin{pmatrix} X_2 & Y_2 \\ Z_2 & -X_2 \end{pmatrix}, (P_2, Q_2, R_2) \right)$ are elements of \mathfrak{g}^J , then the Lie bracket $[\alpha, \beta]$ of α and β is given by

$$(3.1) \quad [\alpha, \beta] = \left(\begin{pmatrix} X^* & Y^* \\ Z^* & -X^* \end{pmatrix}, (P^*, Q^*, R^*) \right),$$

where

$$\begin{aligned} X^* &= X_1 X_2 - X_2 X_1 + Y_1 Z_2 - Y_2 Z_1, \\ Y^* &= X_1 Y_2 - X_2 Y_1 + Y_2 {}^t X_1 - Y_1 {}^t X_2, \\ Z^* &= Z_1 X_2 - Z_2 X_1 + {}^t X_2 Z_1 - {}^t X_1 Z_2, \\ P^* &= P_1 X_2 - P_2 X_1 + Q_1 Z_2 - Q_2 Z_1, \\ Q^* &= P_1 Y_2 - P_2 Y_1 + Q_2 {}^t X_1 - Q_1 {}^t X_2, \\ R^* &= P_1 {}^t Q_2 - P_2 {}^t Q_1 + Q_2 {}^t P_1 - Q_1 {}^t P_2 \end{aligned}$$

Lemma 3.1.

$$[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J, \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J.$$

Proof. The proof follows immediately from Formula (3.1). □

Lemma 3.2. *Let*

$$k^J = \left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, (0, 0, \kappa) \right) \in K^J$$

with $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K$, $\kappa = {}^t \kappa \in \mathbb{R}^{(m,m)}$ and

$$\alpha = \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P, Q, 0) \right) \in \mathfrak{p}^J$$

with $X = {}^t X$, $Y = {}^t Y \in \mathbb{R}^{(n,n)}$, $P, Q \in \mathbb{R}^{(m,n)}$. Then the adjoint action of K^J on \mathfrak{p}^J is given by

$$(3.2) \quad Ad(k^J)\alpha = \left(\begin{pmatrix} X_* & Y_* \\ Y_* & -X_* \end{pmatrix}, (P_*, Q_*, 0) \right),$$

where

$$(3.3) \quad X_* = AX {}^t A - (BX {}^t B + BY {}^t A + AY {}^t B),$$

$$(3.4) \quad Y_* = (AX {}^t B + AY {}^t A + BX {}^t A) - BY {}^t B,$$

$$(3.5) \quad P_* = P {}^t A - Q {}^t B,$$

$$(3.6) \quad Q_* = P {}^t B + Q {}^t A.$$

Proof. We leave the proof to the reader. □

We recall that T_n denotes the vector space of all $n \times n$ symmetric complex matrices. For brevity, we put $T_{n,m} := T_n \times \mathbb{C}^{(m,n)}$. We define the real linear map $\Phi : \mathfrak{p}^J \longrightarrow T_{n,m}$ by

$$(3.7) \quad \Phi \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P, Q, 0) \right) = (X + iY, P + iQ),$$

where $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}$ and $P, Q \in \mathbb{R}^{(m,n)}$.

Let $S(n, \mathbb{R})$ denote the additive group consisting of all $n \times n$ real symmetric matrices. Now we define the isomorphism $\theta : K^J \longrightarrow U(n) \times S(n, \mathbb{R})$ by

$$(3.8) \quad \theta(h, (0, 0, \kappa)) = (\delta(h), \kappa), \quad h \in K, \quad \kappa \in S(n, \mathbb{R}),$$

where $\delta : K \longrightarrow U(n)$ is the map defined by (2.5). Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can identify \mathfrak{p}^J with $T_n \times \mathbb{C}^{(m,n)}$.

Theorem 3.1. *The adjoint representation of K^J on \mathfrak{p}^J is compatible with the natural action of $U(n) \times S(n, \mathbb{R})$ on $T_{n,m}$ defined by*

$$(3.9) \quad (h, \kappa) \cdot (\omega, z) := (h \omega^t h, z^t h), \quad h \in U(n), \quad \kappa \in S(n, \mathbb{R}), \quad (\omega, z) \in T_{n,m}$$

through the maps Φ and θ . Precisely, if $k^J \in K^J$ and $\alpha \in \mathfrak{p}^J$, then we have the following equality

$$(3.10) \quad \Phi(Ad(k^J)\alpha) = \theta(k^J) \cdot \Phi(\alpha).$$

Here we regard the complex vector space $T_{n,m}$ as a real vector space.

Proof. Let

$$k^J = \left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, (0, 0, \kappa) \right) \in K^J$$

with $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K$, $\kappa = {}^t\kappa \in \mathbb{R}^{(m,m)}$ and

$$\alpha = \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P, Q, 0) \right) \in \mathfrak{p}^J$$

with $X = {}^tX$, $Y = {}^tY \in \mathbb{R}^{(n,n)}$, $P, Q \in \mathbb{R}^{(m,n)}$. Then we have

$$\begin{aligned} \theta(k^J) \cdot \Phi(\alpha) &= (A + iB, \kappa) \cdot (X + iY, P + iQ) \\ &= ((A + iB)(X + iY) {}^t(A + iB), (P + iQ) {}^t(A + iB)) \\ &= (X_* + iY_*, P_* + iQ_*) \\ &= \Phi \left(\begin{pmatrix} X_* & Y_* \\ Y_* & -X_* \end{pmatrix}, (P_*, Q_*, 0) \right) \\ &= \Phi(Ad(k^J)\alpha) \quad (\text{by Lemma 3.2}), \end{aligned}$$

where X_*, Y_*, Z_* and Q_* are given by the formulas (3.3), (3.4), (3.5) and (3.6) respectively. \square

We now study the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all differential operators on $\mathbb{H}_{n,m}$ invariant under the *natural action* (1.2) of G^J . The action (3.9) induces the action of $U(n)$ on the polynomial algebra $\text{Pol}_{n,m} := \text{Pol}(T_{n,m})$. We denote by $\text{Pol}_{n,m}^{U(n)}$ the subalgebra of $\text{Pol}_{n,m}$ consisting of all $U(n)$ -invariants. Similarly the action (3.2) of K induces the action of K on the polynomial algebra $\text{Pol}(\mathfrak{p}^J)$. We see that through the identification of \mathfrak{p}^J with $T_{n,m}$, the algebra $\text{Pol}(\mathfrak{p}^J)$ is isomorphic to $\text{Pol}_{n,m}$. The following $U(n)$ -invariant inner product $(\ , \)_*$ of the complex vector space $T_{n,m}$ defined by

$$((\omega, z), (\omega', z'))_* = \text{tr}(\omega \overline{\omega'}) + \text{tr}(z {}^t \overline{z'}), \quad (\omega, z), (\omega', z') \in T_{n,m}$$

gives a canonical isomorphism

$$T_{n,m} \cong T_{n,m}^*, \quad (\omega, z) \mapsto f_{\omega, z}, \quad (\omega, z) \in T_{n,m},$$

where $f_{\omega, z}$ is the linear functional on $T_{n,m}$ defined by

$$f_{\omega, z}((\omega', z')) = ((\omega', z'), (\omega, z))_*, \quad (\omega', z') \in T_{n,m}.$$

According to Helgason ([10], p. 287), one gets a canonical linear bijection of $S(T_{n,m})^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. Identifying $T_{n,m}$ with $T_{n,m}^*$ by the above isomorphism, one gets a natural linear bijection

$$\Theta_{n,m} : \text{Pol}_{n,m}^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m})$$

of $\text{Pol}_{n,m}^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. The map $\Theta_{n,m}$ is described explicitly as follows. We put $N_* = n(n+1) + 2mn$. Let $\{\eta_\alpha \mid 1 \leq \alpha \leq N_*\}$ be a basis of \mathfrak{p}^J . If $P \in \text{Pol}(\mathfrak{p}^J)^K = \text{Pol}_{n,m}^{U(n)}$, then

$$(3.11) \quad (\Theta_{n,m}(P)f)(gK^J) = \left[P \left(\frac{\partial}{\partial t_\alpha} \right) f \left(g \exp \left(\sum_{\alpha=1}^{N_*} t_\alpha \eta_\alpha \right) K^J \right) \right]_{(t_\alpha)=0},$$

where $f \in C^\infty(\mathbb{H}_{n,m})$. In general, it is hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P \in \text{Pol}(\mathfrak{p}^J)^K$. We refer to [10], p. 287.

We present the following *basic* $U(n)$ -invariant polynomials in $\text{Pol}_{n,m}^{U(n)}$.

$$(3.12) \quad q_j(\omega, z) = \text{tr}((\omega \overline{\omega})^{j+1}), \quad 0 \leq j \leq n-1,$$

$$(3.13) \quad \alpha_{kp}^{(j)}(\omega, z) = \text{Re}(z(\overline{\omega}\omega)^j {}^t \overline{z})_{kp}, \quad 0 \leq j \leq n-1, \quad 1 \leq k \leq p \leq m,$$

$$(3.14) \quad \beta_{lq}^{(j)}(\omega, z) = \text{Im}(z(\overline{\omega}\omega)^j {}^t \overline{z})_{lq}, \quad 0 \leq j \leq n-1, \quad 1 \leq l < q \leq m,$$

$$(3.15) \quad f_{kp}^{(j)}(\omega, z) = \text{Re}(z(\overline{\omega}\omega)^j \overline{\omega} {}^t z)_{kp}, \quad 0 \leq j \leq n-1, \quad 1 \leq k \leq p \leq m,$$

$$(3.16) \quad g_{kp}^{(j)}(\omega, z) = \text{Im}(z(\overline{\omega}\omega)^j \overline{\omega} {}^t z)_{kp}, \quad 0 \leq j \leq n-1, \quad 1 \leq k \leq p \leq m,$$

where $\omega \in T_n$ and $z \in \mathbb{C}^{(m,n)}$.

We present some interesting $U(n)$ -invariants. For an $m \times m$ matrix S , we define the following invariant polynomials in $\text{Pol}_{n,m}^{U(n)}$:

$$(3.17) \quad m_{j;S}^{(1)}(\omega, z) = \text{Re} \left(\text{tr}(\omega \bar{\omega} + {}^t z S \bar{z})^j \right), \quad 1 \leq j \leq n,$$

$$(3.18) \quad m_{j;S}^{(2)}(\omega, z) = \text{Im} \left(\text{tr}(\omega \bar{\omega} + {}^t z S \bar{z})^j \right), \quad 1 \leq j \leq n,$$

$$(3.19) \quad q_{k;S}^{(1)}(\omega, z) = \text{Re} \left(\text{tr}({}^t z S \bar{z})^k \right), \quad 1 \leq k \leq m,$$

$$(3.20) \quad q_{k;S}^{(2)}(\omega, z) = \text{Im} \left(\text{tr}({}^t z S \bar{z})^k \right), \quad 1 \leq k \leq m,$$

$$(3.21) \quad \theta_{i,k,j;S}^{(1)}(\omega, z) = \text{Re} \left(\text{tr}((\omega \bar{\omega})^i ({}^t z S \bar{z})^k (\omega \bar{\omega} + {}^t z S \bar{z})^j) \right),$$

$$(3.22) \quad \theta_{i,k,j;S}^{(2)}(\omega, z) = \text{Im} \left(\text{tr}((\omega \bar{\omega})^i ({}^t z S \bar{z})^k (\omega \bar{\omega} + {}^t z S \bar{z})^j) \right),$$

where $1 \leq i, j \leq n$ and $1 \leq k \leq m$.

We define the following $U(n)$ -invariant polynomials in $\text{Pol}_{n,m}^{U(n)}$.

$$(3.23) \quad r_{jk}^{(1)}(\omega, z) = \text{Re} \left(\det((\omega \bar{\omega})^j ({}^t z \bar{z})^k) \right), \quad 1 \leq j \leq n, \quad 1 \leq k \leq m,$$

$$(3.24) \quad r_{jk}^{(2)}(\omega, z) = \text{Im} \left(\det((\omega \bar{\omega})^j ({}^t z \bar{z})^k) \right), \quad 1 \leq j \leq n, \quad 1 \leq k \leq m.$$

We propose the following natural problems.

Problem 1. Find a complete list of explicit generators of $\text{Pol}_{n,m}^{U(n)}$.

Problem 2. Find all the relations among a set of generators of $\text{Pol}_{n,m}^{U(n)}$.

Problem 3. Find an easy or effective way to express the images of the above invariant polynomials or generators of $\text{Pol}_{n,m}^{U(n)}$ under the Helgason map $\Theta_{n,m}$ explicitly.

Problem 4. Decompose $\text{Pol}_{n,m}$ into $U(n)$ -irreducibles.

Problem 5. Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$. Or construct explicit G^J -invariant differential operators on $\mathbb{H}_{n,m}$.

Problem 6. Find all the relations among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.

Problem 7. Is $\text{Pol}_{n,m}^{U(n)}$ finitely generated? Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generated?

Quite recently Minoru Itoh [12] solved Problem 1 and Problem 7.

Theorem 3.2. $\text{Pol}_{n,m}^{U(n)}$ is generated by

$$q_j(\omega, z), \quad \alpha_{kp}^{(j)}(\omega, z), \quad \beta_{lq}^{(j)}(\omega, z), \quad f_{kp}^{(j)}(\omega, z) \quad \text{and} \quad g_{kp}^{(j)}(\omega, z),$$

where $0 \leq j \leq n-1$, $1 \leq k \leq p \leq m$ and $1 \leq l < q \leq m$.

4. Examples of Explicit G^J -Invariant Differential Operators

In this section we give examples of explicit G^J -invariant differential operators on the Siegel-Jacobi space and the Siegel-Jacobi disk.

For $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, we set

$$\begin{aligned}\Omega_* &= M \cdot \Omega = X_* + i Y_*, \quad X_*, Y_* \text{ real}, \\ Z_* &= (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1} = U_* + i V_*, \quad U_*, V_* \text{ real}.\end{aligned}$$

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega$, $d\bar{\Omega}$, $\frac{\partial}{\partial\Omega}$, $\frac{\partial}{\partial\bar{\Omega}}$ as before and set

$$\begin{aligned}Z &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real}, \\ dZ &= (dz_{kl}), \quad d\bar{Z} = (d\bar{z}_{kl}),\end{aligned}$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \bar{Z}} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_{11}} & \cdots & \frac{\partial}{\partial \bar{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \bar{z}_{1n}} & \cdots & \frac{\partial}{\partial \bar{z}_{mn}} \end{pmatrix}.$$

Then we can show that

$$(4.1) \quad d\Omega_* = {}^t(C\Omega + D)^{-1} d\Omega (C\Omega + D)^{-1},$$

$$(4.2) \quad dZ_* = dZ(C\Omega + D)^{-1} + \{\lambda - (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}C\} d\Omega (C\Omega + D)^{-1},$$

$$(4.3) \quad \begin{aligned} \frac{\partial}{\partial \Omega_*} &= (C\Omega + D) \left\{ (C\Omega + D) \frac{\partial}{\partial \Omega} \right\} \\ &\quad + (C\Omega + D) \left\{ (C {}^t Z + C {}^t \mu - D {}^t \lambda) \left(\frac{\partial}{\partial Z} \right) \right\} \end{aligned}$$

and

$$(4.4) \quad \frac{\partial}{\partial Z_*} = (C\Omega + D) \frac{\partial}{\partial Z}.$$

From [14, p. 33] or [20, p. 128], we know that

$$(4.5) \quad Y_* = {}^t(C\bar{\Omega} + D)^{-1} Y (C\Omega + D)^{-1} = {}^t(C\Omega + D)^{-1} Y (C\bar{\Omega} + D)^{-1}.$$

Using Formulas (4.1), (4.2) and (4.5), the author [29] proved that for any two positive real numbers A and B ,

$$\begin{aligned} ds_{n,m;A,B}^2 = & A \operatorname{tr} \left(Y^{-1} d\Omega Y^{-1} d\overline{\Omega} \right) \\ & + B \left\{ \operatorname{tr} \left(Y^{-1} {}^t V V Y^{-1} d\Omega Y^{-1} d\overline{\Omega} \right) + \operatorname{tr} \left(Y^{-1} {}^t (dZ) d\overline{Z} \right) \right. \\ & \left. - \operatorname{tr} \left(V Y^{-1} d\Omega Y^{-1} {}^t (d\overline{Z}) \right) - \operatorname{tr} \left(V Y^{-1} d\overline{\Omega} Y^{-1} {}^t (dZ) \right) \right\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of G^J .

The following lemma is very useful for computing the invariant differential operators. H. Maass [13] observed the following useful fact.

Lemma 4.1. (a) Let A be an $m \times n$ matrix and B an $n \times l$ matrix. Assume that the entries of A commute with the entries of B . Then ${}^t(AB) = {}^tB {}^tA$.

(b) Let A , B and C be a $k \times l$, an $n \times m$ and an $m \times l$ matrix respectively. Assume that the entries of A commute with the entries of B . Then

$${}^t(A {}^t(BC)) = B {}^t(A {}^tC).$$

Proof. The proof follows immediately from the direct computation. \square

Using Formulas (4.3), (4.4), (4.5) and Lemma 4.1, the author [29] proved that the following differential operators \mathbb{M}_1 and \mathbb{M}_2 on $\mathbb{H}_{n,m}$ defined by

$$(4.6) \quad \mathbb{M}_1 = \operatorname{tr} \left(Y \frac{\partial}{\partial Z} {}^t \left(\frac{\partial}{\partial \overline{Z}} \right) \right)$$

and

$$(4.7) \quad \begin{aligned} \mathbb{M}_2 = & \operatorname{tr} \left(Y {}^t \left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) + \operatorname{tr} \left(V Y^{-1} {}^t V {}^t \left(Y \frac{\partial}{\partial \overline{Z}} \right) \frac{\partial}{\partial Z} \right) \\ & + \operatorname{tr} \left(V {}^t \left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial Z} \right) + \operatorname{tr} \left({}^t V {}^t \left(Y \frac{\partial}{\partial \overline{Z}} \right) \frac{\partial}{\partial \Omega} \right) \end{aligned}$$

are invariant under the action (1.2) of G^J . The author [29] proved that for any two positive real numbers A and B , the following differential operator

$$(4.8) \quad \Delta_{n,m;A,B} = \frac{4}{A} \mathbb{M}_2 + \frac{4}{B} \mathbb{M}_1$$

is the Laplacian of the G^J -invariant Riemannian metric $ds_{n,m;A,B}^2$.

Proposition 4.1. *The following differential operator \mathbb{K} on $\mathbb{H}_{n,m}$ of degree $2n$ defined by*

$$(4.9) \quad \mathbb{K} = \det(Y) \det \left(\frac{\partial}{\partial Z} {}^t \left(\frac{\partial}{\partial \bar{Z}} \right) \right)$$

is invariant under the action (1.2) of G^J .

Proof. Let $\mathbb{K}_{M,(\lambda,\mu;\kappa)}$ denote the image of \mathbb{K} under the transformation

$$(\Omega, Z) \mapsto ((M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1})$$

with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$. If f is a C^∞ function on $\mathbb{H}_{n,m}$, using (4.4), (4.5) and Lemma 4.1, we have

$$\begin{aligned} \mathbb{K}_{M,(\lambda,\mu;\kappa)} f &= \det(Y) |\det(C\Omega + D)|^{-2} \det \left[(C\Omega + D) \frac{\partial}{\partial Z} {}^t \left\{ (C\bar{\Omega} + D) \frac{\partial f}{\partial \bar{Z}} \right\} \right] \\ &= \det(Y) |\det(C\Omega + D)|^{-2} \det \left[(C\Omega + D) {}^t \left\{ (C\bar{\Omega} + D) {}^t \left(\frac{\partial}{\partial Z} {}^t \left(\frac{\partial f}{\partial \bar{Z}} \right) \right) \right\} \right] \\ &= \det(Y) |\det(C\Omega + D)|^{-2} \det \left[(C\Omega + D) \frac{\partial}{\partial Z} {}^t \left(\frac{\partial f}{\partial \bar{Z}} \right) {}^t (C\bar{\Omega} + D) \right] \\ &= \det(Y) \det \left(\frac{\partial}{\partial Z} {}^t \left(\frac{\partial f}{\partial \bar{Z}} \right) \right) \\ &= \mathbb{K} f. \end{aligned}$$

Since $M \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ are arbitrary, \mathbb{K} is invariant under the action (1.2) of G^J . \square

Proposition 4.2. *The following matrix-valued differential operator \mathbb{T} on $\mathbb{H}_{n,m}$ defined by*

$$(4.10) \quad \mathbb{T} = {}^t \left(\frac{\partial}{\partial \bar{Z}} \right) Y \frac{\partial}{\partial Z}$$

is invariant under the action (1.2) of G^J .

Proof. Let $\mathbb{T}_{M,(\lambda,\mu;\kappa)}$ denote the image of \mathbb{T} under the transformation

$$(\Omega, Z) \mapsto ((M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1})$$

with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$. If f is a C^∞ function on $\mathbb{H}_{n,m}$, according to (4.4), (4.5) and Lemma 4.1, we have

$$\begin{aligned} \mathbb{T}_{M,(\lambda,\mu;\kappa)} f &= {}^t \left((C\overline{\Omega} + D) \frac{\partial}{\partial \overline{Z}} \right) {}^t (C\overline{\Omega} + D)^{-1} Y (C\Omega + D)^{-1} (C\Omega + D) \frac{\partial f}{\partial Z} \\ &= {}^t \left(\frac{\partial}{\partial \overline{Z}} \right) Y \frac{\partial f}{\partial Z} \\ &= \mathbb{T}f. \end{aligned}$$

Since $M \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ are arbitrary, \mathbb{T} is invariant under the action (1.2) of G^J . \square

Corollary 4.1. *Each (k, l) -entry \mathbb{T}_{kl} of \mathbb{T} given by*

$$(4.11) \quad \mathbb{T}_{kl} = \sum_{i,j=1}^n y_{ij} \frac{\partial^2}{\partial \overline{z}_{ki} \partial z_{lj}}, \quad 1 \leq k, l \leq m$$

is an element of $\mathbb{D}(\mathbb{H}_{n,m})$.

Proof. It follows immediately from Proposition 4.2. \square

Now we consider invariant differential operators on the Siegel-Jacobi disk. Let

$$\mathbb{D}_n = \{ W \in \mathbb{C}^{(n,n)} \mid W = {}^t W, I_n - \overline{W}W > 0 \}$$

be the generalized unit disk.

For brevity, we write $\mathbb{D}_{n,m} := \mathbb{D}_n \times \mathbb{C}^{(m,n)}$. For a coordinate $(W, \eta) \in \mathbb{D}_{n,m}$ with $W = (w_{\mu\nu}) \in \mathbb{D}_n$ and $\eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned} dW &= (dw_{\mu\nu}), & d\overline{W} &= (d\overline{w}_{\mu\nu}), \\ d\eta &= (d\eta_{kl}), & d\overline{\eta} &= (d\overline{\eta}_{kl}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial W} &= \left(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial w_{\mu\nu}} \right), & \frac{\partial}{\partial \overline{W}} &= \left(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{w}_{\mu\nu}} \right), \\ \frac{\partial}{\partial \eta} &= \begin{pmatrix} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \eta_{1n}} & \cdots & \frac{\partial}{\partial \eta_{mn}} \end{pmatrix}, & \frac{\partial}{\partial \overline{\eta}} &= \begin{pmatrix} \frac{\partial}{\partial \overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{\eta}_{1n}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{mn}} \end{pmatrix}. \end{aligned}$$

We can identify an element $g = (M, (\lambda, \mu; \kappa))$ of G^J , $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ with the element

$$\begin{pmatrix} A & 0 & B & A^t \mu - B^t \lambda \\ \lambda & I_m & \mu & \kappa \\ C & 0 & D & C^t \mu - D^t \lambda \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

of $Sp(m+n, \mathbb{R})$.

We set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n} \end{pmatrix}.$$

We now consider the group G_*^J defined by

$$G_*^J := T_*^{-1} G^J T_*.$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$(4.12) \quad T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix},$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2} \{Q^t (\lambda + i\mu) - P^t (\lambda - i\mu)\} \\ \frac{1}{2} (\lambda + i\mu) & I_h + i\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2} \{P^t (\lambda - i\mu) - Q^t (\lambda + i\mu)\} \\ \frac{1}{2} (\lambda - i\mu) & -i\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by the formulas

$$(4.13) \quad P = \frac{1}{2} \{(A + D) + i(B - C)\}$$

and

$$(4.14) \quad Q = \frac{1}{2} \{(A - D) - i(B + C)\}.$$

From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2} (\lambda + i\mu), \frac{1}{2} (\lambda - i\mu); -i\frac{\kappa}{2} \right) \right) := \left(\begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix} \right).$$

In other words, we have the relation

$$T_*^{-1} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2} (\lambda + i\mu), \frac{1}{2} (\lambda - i\mu); -i\frac{\kappa}{2} \right) \right).$$

Let

$$H_{\mathbb{C}}^{(n,m)} := \{(\xi, \eta; \zeta) \mid \xi, \eta \in \mathbb{C}^{(m,n)}, \zeta \in \mathbb{C}^{(m,m)}, \zeta + \eta^t \xi \text{ symmetric}\}$$

be the complex Heisenberg group endowed with the following multiplication

$$(\xi, \eta; \zeta) \circ (\xi', \eta'; \zeta') := (\xi + \xi', \eta + \eta'; \zeta + \zeta' + \xi^t \eta' - \eta^t \xi').$$

We define the semidirect product

$$SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$$

endowed with the following multiplication

$$\begin{aligned} & \left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi, \eta; \zeta) \right) \cdot \left(\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi', \eta'; \zeta') \right) \\ &= \left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\tilde{\xi} + \xi', \tilde{\eta} + \eta'; \zeta + \zeta' + \tilde{\xi}^t \eta' - \tilde{\eta}^t \xi') \right), \end{aligned}$$

where $\tilde{\xi} = \xi P' + \eta R'$ and $\tilde{\eta} = \xi Q' + \eta S'$.

If we identify $H_{\mathbb{R}}^{(n,m)}$ with the subgroup

$$\{(\xi, \bar{\xi}; i\kappa) \mid \xi \in \mathbb{C}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}\}$$

of $H_{\mathbb{C}}^{(n,m)}$, we have the following inclusion

$$G_*^J \subset SU(n, n) \ltimes H_{\mathbb{R}}^{(n,m)} \subset SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}.$$

We define the mapping $\Theta : G^J \longrightarrow G_*^J$ by

$$(4.15) \quad \Theta \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) := \left(\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu), \frac{1}{2}(\lambda - i\mu); -i\frac{\kappa}{2} \right) \right),$$

where P and Q are given by (4.13) and (4.14). We can see that if $g_1, g_2 \in G^J$, then $\Theta(g_1 g_2) = \Theta(g_1) \Theta(g_2)$.

According to [26, p. 250], G_*^J is of the Harish-Chandra type (cf. [17, p. 118]). Let

$$g_* = \left(\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, (\lambda, \mu; \kappa) \right)$$

be an element of G_*^J . Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ in $SU(n, n)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n \end{pmatrix},$$

the P_*^+ -component of the following element

$$g_* \cdot \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right), \quad W \in \mathbb{D}_n$$

of $SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$ is given by

$$(4.16) \quad \left(\begin{pmatrix} I_n & (PW + Q)(\bar{Q}W + \bar{P})^{-1} \\ 0 & I_n \end{pmatrix}, (0, (\eta + \lambda W + \mu)(\bar{Q}W + \bar{P})^{-1}; 0) \right).$$

We can identify $\mathbb{D}_{n,m}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \mid W \in \mathbb{D}_n, \eta \in \mathbb{C}^{(m,n)} \right\}$$

of the complexification of G_*^J . Indeed, $\mathbb{D}_{n,m}$ is embedded into P_*^+ given by

$$P_*^+ = \left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \mid W = {}^t W \in \mathbb{C}^{(n,n)}, \eta \in \mathbb{C}^{(m,n)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (cf. [17, p. 119]). Then we get the *natural transitive action* of G_*^J on $\mathbb{D}_{n,m}$ defined by

$$(4.17) \quad \begin{aligned} & \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \bar{\xi}; i\kappa) \right) \cdot (W, \eta) \\ &= \left((PW + Q)(\overline{Q}W + \overline{P})^{-1}, (\eta + \xi W + \bar{\xi})(\overline{Q}W + \overline{P})^{-1} \right), \end{aligned}$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$, $\xi \in \mathbb{C}^{(m,n)}$, $\kappa \in \mathbb{R}^{(m,m)}$ and $(W, \eta) \in \mathbb{D}_{n,m}$.

The author [30] proved that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (4.17) of G_*^J on $\mathbb{D}_{n,m}$ through a *partial Cayley transform* $\Phi : \mathbb{D}_{n,m} \longrightarrow \mathbb{H}_{n,m}$ defined by

$$(4.18) \quad \Phi(W, \eta) := \left(i(I_n + W)(I_n - W)^{-1}, 2i\eta(I_n - W)^{-1} \right).$$

In other words, if $g_0 \in G^J$ and $(W, \eta) \in \mathbb{D}_{n,m}$,

$$(4.19) \quad g_0 \cdot \Phi(W, \eta) = \Phi(g_* \cdot (W, \eta)),$$

where $g_* = T_*^{-1}g_0T_*$. Φ is a biholomorphic mapping of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ which gives the partially bounded realization of $\mathbb{H}_{n,m}$ by $\mathbb{D}_{n,m}$. The inverse of Φ is

$$\Phi^{-1}(\Omega, Z) = \left((\Omega - iI_n)(\Omega + iI_n)^{-1}, Z(\Omega + iI_n)^{-1} \right).$$

For $(W, \eta) \in \mathbb{D}_{n,m}$, we write

$$(\Omega, Z) := \Phi(W, \eta).$$

Thus

$$(4.20) \quad \Omega = i(I_n + W)(I_n - W)^{-1}, \quad Z = 2i\eta(I_n - W)^{-1}.$$

Since

$$d(I_n - W)^{-1} = (I_n - W)^{-1}dW(I_n - W)^{-1}$$

and

$$I_n + (I_n + W)(I_n - W)^{-1} = 2(I_n - W)^{-1},$$

we get the following formulas from (4.20)

$$(4.21) \quad Y = \frac{1}{2i} (\Omega - \bar{\Omega}) = (I_n - W)^{-1} (I_n - W\bar{W}) (I_n - \bar{W})^{-1},$$

$$(4.22) \quad V = \frac{1}{2i} (Z - \bar{Z}) = \eta (I_n - W)^{-1} + \bar{\eta} (I_n - \bar{W})^{-1},$$

$$(4.23) \quad d\Omega = 2i (I_n - W)^{-1} dW (I_n - W)^{-1},$$

$$(4.24) \quad dZ = 2i \left\{ d\eta + \eta (I_n - W)^{-1} dW \right\} (I_n - W)^{-1}.$$

Using Formulas (4.18), (4.20)-(4.24), the author [31] proved that for any two positive real numbers A and B , the following metric $d\tilde{s}_{n,m;A,B}^2$ defined by

$$\begin{aligned} ds_{\mathbb{D}_{n,m;A,B}}^2 = & 4A \operatorname{tr} \left((I_n - W\bar{W})^{-1} dW (I_n - \bar{W}W)^{-1} d\bar{W} \right) \\ & + 4B \left\{ \operatorname{tr} \left((I_n - W\bar{W})^{-1} {}^t(d\eta) \beta \right) \right. \\ & + \operatorname{tr} \left((\eta\bar{W} - \bar{\eta})(I_n - W\bar{W})^{-1} dW (I_n - \bar{W}W)^{-1} {}^t(d\bar{\eta}) \right) \\ & + \operatorname{tr} \left((\bar{\eta}W - \eta)(I_n - \bar{W}W)^{-1} d\bar{W} (I_n - W\bar{W})^{-1} {}^t(d\eta) \right) \\ & - \operatorname{tr} \left((I_n - W\bar{W})^{-1} {}^t\eta \eta (I_n - \bar{W}W)^{-1} \bar{W} dW (I_n - \bar{W}W)^{-1} d\bar{W} \right) \\ & - \operatorname{tr} \left(W (I_n - \bar{W}W)^{-1} {}^t\bar{\eta} \bar{\eta} (I_n - W\bar{W})^{-1} dW (I_n - \bar{W}W)^{-1} d\bar{W} \right) \\ & + \operatorname{tr} \left((I_n - W\bar{W})^{-1} {}^t\eta \bar{\eta} (I_n - W\bar{W})^{-1} dW (I_n - \bar{W}W)^{-1} d\bar{W} \right) \\ & + \operatorname{tr} \left((I_n - \bar{W})^{-1} {}^t\bar{\eta} \eta \bar{W} (I_n - W\bar{W})^{-1} dW (I_n - \bar{W}W)^{-1} d\bar{W} \right) \\ & + \operatorname{tr} \left((I_n - \bar{W})^{-1} (I_n - W) (I_n - \bar{W}W)^{-1} {}^t\bar{\eta} \eta (I_n - \bar{W}W)^{-1} \right. \\ & \quad \times (I_n - \bar{W}) (I_n - W)^{-1} dW (I_n - \bar{W}W)^{-1} d\bar{W} \left. \right) \\ & \left. - \operatorname{tr} \left((I_n - W\bar{W})^{-1} (I_n - W) (I_n - \bar{W})^{-1} {}^t\bar{\eta} \eta (I_n - W)^{-1} \right. \right. \\ & \quad \left. \left. \times dW (I_n - \bar{W}W)^{-1} d\bar{W} \right) \right\} \end{aligned}$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ which is invariant under the action (4.17) of the Jacobi group G_*^J .

We note that if $n = m = 1$ and $A = B = 1$, we get

$$\begin{aligned} \frac{1}{4} ds_{\mathbb{D}_{1,1;1,1}}^2 &= \frac{dW d\bar{W}}{(1 - |W|^2)^2} + \frac{1}{(1 - |W|^2)} d\eta d\bar{\eta} \\ &\quad + \frac{(1 + |W|^2)|\eta|^2 - \bar{W}\eta^2 - W\bar{\eta}^2}{(1 - |W|^2)^3} dW d\bar{W} \\ &\quad + \frac{\eta\bar{W} - \bar{\eta}}{(1 - |W|^2)^2} dW d\bar{\eta} + \frac{\bar{\eta}W - \eta}{(1 - |W|^2)^2} d\bar{W} d\eta. \end{aligned}$$

From the formulas (4.20), (4.23) and (4.24), we get

$$(4.25) \quad \frac{\partial}{\partial \Omega} = \frac{1}{2i} (I_n - W) \left[{}^t \left\{ (I_n - W) \frac{\partial}{\partial W} \right\} - {}^t \left\{ {}^t \eta \left(\frac{\partial}{\partial \eta} \right) \right\} \right]$$

and

$$(4.26) \quad \frac{\partial}{\partial Z} = \frac{1}{2i} (I_n - W) \frac{\partial}{\partial \eta}.$$

Using Formulas (4.20)-(4.22), (4.25), (4.26) and Lemma 4.1, the author [31] proved that the following differential operators \mathbb{S}_1 and \mathbb{S}_2 on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{S}_1 = \sigma \left((I_n - \bar{W}W) \frac{\partial}{\partial \eta} \left(\frac{\partial}{\partial \bar{\eta}} \right) \right)$$

and

$$\begin{aligned} \mathbb{S}_2 &= \text{tr} \left((I_n - W\bar{W}) \left((I_n - W\bar{W}) \frac{\partial}{\partial \bar{W}} \right) \frac{\partial}{\partial W} \right) \\ &\quad + \text{tr} \left({}^t(\eta - \bar{\eta}W) \left(\frac{\partial}{\partial \bar{\eta}} \right) (I_n - \bar{W}W) \frac{\partial}{\partial W} \right) \\ &\quad + \text{tr} \left((\bar{\eta} - \eta\bar{W}) \left((I_n - W\bar{W}) \frac{\partial}{\partial \bar{W}} \right) \frac{\partial}{\partial \eta} \right) \\ &\quad - \text{tr} \left(\eta\bar{W}(I_n - W\bar{W})^{-1} {}^t \eta \left(\frac{\partial}{\partial \bar{\eta}} \right) (I_n - \bar{W}W) \frac{\partial}{\partial \eta} \right) \\ &\quad - \text{tr} \left(\bar{\eta}W(I_n - \bar{W}W)^{-1} {}^t \bar{\eta} \left(\frac{\partial}{\partial \bar{\eta}} \right) (I_n - \bar{W}W) \frac{\partial}{\partial \eta} \right) \\ &\quad + \text{tr} \left(\bar{\eta}(I_n - W\bar{W})^{-1} {}^t \eta \left(\frac{\partial}{\partial \bar{\eta}} \right) (I_n - \bar{W}W) \frac{\partial}{\partial \eta} \right) \\ &\quad + \text{tr} \left(\eta\bar{W}W(I_n - \bar{W}W)^{-1} {}^t \bar{\eta} \left(\frac{\partial}{\partial \bar{\eta}} \right) (I_n - \bar{W}W) \frac{\partial}{\partial \eta} \right) \end{aligned}$$

are invariant under the action (4.17) of G_*^J . The author also proved that

$$(4.27) \quad \Delta_{\mathbb{D}_{n,m};A,B} := \frac{1}{A} \mathbb{S}_2 + \frac{1}{B} \mathbb{S}_1$$

is the Laplacian of the invariant metric $ds_{\mathbb{D}_{n,m};A,B}^2$ on $\mathbb{D}_{n,m}$ (cf. [31]).

Proposition 4.3. *The following differential operator on $\mathbb{D}_{n,m}$ defined by*

$$(4.28) \quad \mathbb{K}_{\mathbb{D}} = \det(I_n - \overline{W}W) \det \left(\frac{\partial}{\partial \eta} {}^t \left(\frac{\partial}{\partial \overline{\eta}} \right) \right)$$

is invariant under the action (4.17) of G_^J on $\mathbb{D}_{n,m}$.*

Proof. It follows from Proposition 4.1, Formulas (4.21), (4.26) and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (4.17) of G_*^J on $\mathbb{D}_{n,m}$ via the partial Cayley transform. \square

Proposition 4.4. *The following matrix-valued differential operator on $\mathbb{D}_{n,m}$ defined by*

$$(4.29) \quad \mathbb{T}^{\mathbb{D}} := \left(\frac{\partial}{\partial \overline{\eta}} \right) (I_n - \overline{W}W) \frac{\partial}{\partial \eta}$$

is invariant under the action (4.17) of G_^J on $\mathbb{D}_{n,m}$.*

Proof. It follows from Proposition 4.2, Formulas (4.21), (4.26) and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (4.17) of G_*^J on $\mathbb{D}_{n,m}$ via the partial Cayley transform. \square

Corollary 4.2. *Each (k, l) -entry $\mathbb{T}_{kl}^{\mathbb{D}}$ of $\mathbb{T}^{\mathbb{D}}$ given by*

$$(4.30) \quad \mathbb{T}_{kl}^{\mathbb{D}} = \sum_{i,j=1}^n \left(\delta_{ij} - \sum_{r=1}^n \overline{w}_{ir} w_{jr} \right) \frac{\partial^2}{\partial \overline{\eta}_{ki} \partial \eta_{lj}}, \quad 1 \leq k, l \leq m$$

is a G_^J -invariant differential operator on $\mathbb{D}_{n,m}$.*

Proof. It follows immediately from Proposition 4.4. \square

For two differential operators D_1 and D_2 on $\mathbb{H}_{n,m}$ or $\mathbb{D}_{n,m}$, we write

$$[D_1, D_2] := D_1 D_2 - D_2 D_1.$$

Then

$$(4.31) \quad \mathbb{M}_3 = [\mathbb{M}_1, \mathbb{M}_2] = \mathbb{M}_1 \mathbb{M}_2 - \mathbb{M}_2 \mathbb{M}_1$$

is an invariant differential operator of degree three on $\mathbb{H}_{n,m}$ and

$$(4.32) \quad \mathbb{P}_{kl} = [\mathbb{K}, \mathbb{T}_{kl}] = \mathbb{K} \mathbb{T}_{kl} - \mathbb{T}_{kl} \mathbb{K}, \quad 1 \leq k, l \leq m$$

is an invariant differential operator of degree $2n + 1$ on $\mathbb{H}_{n,m}$.

Similarly

$$(4.33) \quad \mathbb{S}_3 = [\mathbb{S}_1, \mathbb{S}_2] = \mathbb{S}_1 \mathbb{S}_2 - \mathbb{S}_2 \mathbb{S}_1$$

is an invariant differential operator of degree three on $\mathbb{D}_{n,m}$ and

$$(4.34) \quad \mathbb{Q}_{kl} = [\mathbb{K}_{\mathbb{D}}, \mathbb{T}_{kl}^{\mathbb{D}}] = \mathbb{K}_{\mathbb{D}} \mathbb{T}_{kl}^{\mathbb{D}} - \mathbb{T}_{kl}^{\mathbb{D}} \mathbb{K}_{\mathbb{D}}, \quad 1 \leq k, l \leq m$$

is an invariant differential operator of degree $2n + 1$ on $\mathbb{D}_{n,m}$.

Indeed it is very complicated and difficult at this moment to express the generators of the algebra of all G_*^J -invariant differential operators on $\mathbb{D}_{n,m}$ explicitly.

5. The Case $n = m = 1$

We consider the case $n = m = 1$. For a coordinate (ω, z) in $T_{1,1}$, we write $\omega = x + iy$, $z = u + iv$, x, y, u, v real. The author [27] proved that the algebra $\text{Pol}_{1,1}^{U(1)}$ is generated by

$$\begin{aligned} q(\omega, z) &= \frac{1}{4} \omega \bar{\omega} = \frac{1}{4} (x^2 + y^2), \\ \xi(\omega, z) &= z \bar{z} = u^2 + v^2, \\ \phi(\omega, z) &= \frac{1}{2} \text{Re}(z^2 \bar{\omega}) = \frac{1}{2} (u^2 - v^2)x + uv y, \\ \psi(\omega, z) &= \frac{1}{2} \text{Im}(z^2 \bar{\omega}) = \frac{1}{2} (v^2 - u^2)y + uv x. \end{aligned}$$

In [27], using Formula (3.11) the author calculated explicitly the images

$$D_1 = \Theta_{1,1}(q), \quad D_2 = \Theta_{1,1}(\xi), \quad D_3 = \Theta_{1,1}(\phi) \quad \text{and} \quad D_4 = \Theta_{1,1}(\psi)$$

of q , ξ , ϕ and ψ under the Halgason map $\Theta_{1,1}$. We can show that the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is generated by the following differential operators

$$\begin{aligned} D_1 &= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ &\quad + 2yv \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right), \\ D_2 &= y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right), \\ D_3 &= y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} \\ &\quad - \left(v \frac{\partial}{\partial v} + 1 \right) D_2 \end{aligned}$$

and

$$D_4 = y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} D_2,$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v . Moreover, we have

$$D_1 D_2 - D_2 D_1 = 2y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 4y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left(v \frac{\partial}{\partial v} D_2 + D_2 \right).$$

In particular, the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is not commutative. We refer to [1, 27] for more detail.

Recently Hiroyuki Ochiai [15] proved the following results.

Theorem 5.1. *We have the following relation*

$$(5.1) \quad \phi^2 + \psi^2 = q \xi^2.$$

This relation exhausts all the relations among the generators q, ξ, ϕ and ψ of $Pol_{1,1}^{U(1)}$.

Theorem 5.2. *We have the following relations*

- (a) $[D_1, D_2] = 2D_3$
- (b) $[D_1, D_3] = 2D_1 D_2 - 2D_3$
- (c) $[D_2, D_3] = -D_2^2$
- (d) $[D_4, D_1] = 0$
- (e) $[D_4, D_2] = 0$
- (f) $[D_4, D_3] = 0$
- (g) $D_3^2 + D_4^2 = D_2 D_1 D_2$

These seven relations exhaust all the relations among the generators D_1, D_2, D_3 and D_4 of $\mathbb{D}(\mathbb{H}_{1,1})$.

We can prove the following

Theorem 5.3. *The action of $U(1)$ on $Pol_{1,1}^{U(1)}$ is not multiplicity-free.*

Finally we see that for the case $n = m = 1$, the seven problems proposed in Section 3 are completely solved.

Remark 5.1. According to Theorem 5.2, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H}_{1,1})$. We observe that the Laplacian

$$\Delta_{1,1;A,B} = \frac{4}{A} D_1 + \frac{4}{B} D_2 \quad (\text{see (4.8)})$$

of $(\mathbb{H}_{1,1}, ds_{1,1;A,B}^2)$ does not belong to the center of $\mathbb{D}(\mathbb{H}_{1,1})$.

6. The Case $n = 1$ and m is arbitrary

Conley and Raum [5] found the $2m^2 + m + 1$ explicit generators of $\mathbb{D}(\mathbb{H}_{1,m})$ and the explicit one generator of the center of $\mathbb{D}(\mathbb{H}_{1,m})$. They also found the generators of the center of the universal enveloping algebra of $\mathfrak{U}(\mathfrak{g}^J)$ of the Jacobi Lie algebra \mathfrak{g}^J . The number of generators of the center of $\mathfrak{U}(\mathfrak{g}^J)$ is $1 + \frac{m(m+1)}{2}$.

According to Theorem 3.2, $\text{Pol}_{1,m}^{U(1)}$ is generated by

$$(6.1) \quad q(\omega, z) = \text{tr}(\omega \bar{\omega}),$$

$$(6.2) \quad \alpha_{kp}(\omega, z) = \text{Re}(z^t \bar{z})_{kp} = \text{Re}(z_k \bar{z}_p), \quad 1 \leq k \leq p \leq m,$$

$$(6.3) \quad \beta_{lq}(\omega, z) = \text{Im}(z^t \bar{z})_{lq} = \text{Im}(z_l \bar{z}_q), \quad 1 \leq l < q \leq m,$$

$$(6.4) \quad f_{kp}(\omega, z) = \text{Re}(z \bar{\omega}^t z)_{kp} = \text{Re}(\bar{\omega} z_k z_p), \quad 1 \leq k \leq p \leq m,$$

$$(6.5) \quad g_{kp}(\omega, z) = \text{Im}(z \bar{\omega}^t z)_{kp} = \text{Im}(\bar{\omega} z_k z_p), \quad 1 \leq k \leq p \leq m,$$

where $\omega \in T_1$ and $z \in \mathbb{C}^m$.

We let

$$\omega = x + iy \in \mathbb{C} \quad \text{and} \quad z = {}^t(z_1, \dots, z_m) \in \mathbb{C}^m \quad \text{with} \quad z_k = u_k + iv_k, \quad 1 \leq k \leq m,$$

where $x, y, u_1, v_1, \dots, u_m, v_m$ are real. The invariants $q, \alpha_{kp}, \beta_{lq}, f_{kp}$ and g_{kp} are expressed in terms of x, y, u_k, v_l ($1 \leq k, l \leq m$) as follows:

$$\begin{aligned} q(\omega, z) &= x^2 + y^2, \\ \alpha_{kp}(\omega, z) &= u_k u_p + v_k v_p, \quad 1 \leq k \leq p \leq m, \\ \beta_{lq}(\omega, z) &= u_q v_l - u_l v_q, \quad 1 \leq l < q \leq m, \\ f_{kp}(\omega, z) &= x(u_k u_p - v_k v_p) + y(u_k v_p + v_k u_p), \quad 1 \leq k \leq p \leq m, \\ g_{kp}(\omega, z) &= x(u_k v_p + v_k u_p) - y(u_k u_p - v_k v_p), \quad 1 \leq k \leq p \leq m. \end{aligned}$$

Theorem 6.1. The $1 + \frac{m(m+1)}{2}$ relations

$$(6.6) \quad f_{kp}^2 + g_{kp}^2 = q \alpha_{kk} \alpha_{pp}, \quad 1 \leq k \leq p \leq m$$

exhaust all the relations among a set of generators $q, \alpha_{kp}, \beta_{lq}, f_{kp}$ and g_{kp} with $1 \leq k \leq p \leq m$ and $1 \leq l < q \leq m$.

Theorem 6.2. *The action of $U(1)$ on $Pol_{1,m}$ is not multiplicity-free. In fact, if*

$$Pol_{1,m} = \sum_{\sigma \in \widehat{U(1)}} m_{\sigma} \sigma,$$

then $m_{\sigma} = \infty$.

Problem 1, Problem 2, Problem 4, Problem 5 and Problem 7 were solved. Problem 3 can be handled. Finally Problem 6 is unsolved in the case that $n = 1$ and m is arbitrary.

7. Final Remarks

Using G^J -invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

Definition 7.1. *Let*

$$\Gamma_{n,m} := Sp(n, \mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J , where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ are integral} \right\}.$$

*A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a **Maass-Jacobi form** on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3):*

- (MJ1) *f is invariant under $\Gamma_{n,m}$.*
- (MJ2) *f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. Formula (4.8)).*
- (MJ3) *f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that*

$$|f(X + iY, Z)| \leq C |p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

Remark 7.1. *Let \mathbb{D}_* be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to \mathbb{D}_* if f satisfies the conditions (MJ1), (MJ2)* and (MJ3): the condition (MJ2)* is given by*

(MJ2) f is an eigenfunction of any invariant differential operator in \mathbb{D}_* .*

It is natural to propose the following problems.

Problem A: Find all the eigenfunctions of $\Delta_{n,m;A,B}$.

Problem B: Construct Maass-Jacobi forms.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{n,m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H}_{n,m}$ in the usual way defined by

$$(7.1) \quad f_\phi(\Omega, Z) := \sum_{\gamma \in \Gamma_{n,m}^\infty \backslash \Gamma_{n,m}} \phi(\gamma \cdot (\Omega, Z)),$$

where

$$\Gamma_{n,m}^\infty = \left\{ \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{n,m} \mid C = 0 \right\}$$

is a subgroup of $\Gamma_{n,m}$.

We consider the simple case $n = m = 1$ and $A = B = 1$. A metric $ds_{1,1,1}^2$ on $\mathbb{H}_{1,1}$ given by

$$\begin{aligned} ds_{1,1,1}^2 &= \frac{y + v^2}{y^3} (dx^2 + dy^2) + \frac{1}{y} (du^2 + dv^2) \\ &\quad - \frac{2v}{y^2} (dx du + dy dv) \end{aligned}$$

is a G^J -invariant Kähler metric on $\mathbb{H}_{1,1}$. Its Laplacian $\Delta_{1,1,1}$ is given by

$$\begin{aligned} \Delta_{1,1,1} &= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \\ &\quad + (y + v^2) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ &\quad + 2yv \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right). \end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1,1,1}$.

(1) $h(x, y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y) e^{2\pi i a x}$ ($s \in \mathbb{C}$, $a \neq 0$) with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2} \int_0^\infty \exp \left\{ -\frac{z}{2}(t + t^{-1}) \right\} t^{s-1} dt,$$

where $\operatorname{Re} z > 0$.

(2) $y^s, y^s x, y^s u$ ($s \in \mathbb{C}$) with eigenvalue $s(s-1)$.

(3) $y^s v, y^s uv, y^s xv$ with eigenvalue $s(s+1)$.

(4) x, y, u, v, xv, uv with eigenvalue 0.

(5) All Maass wave forms.

Let ρ be a rational representation of $GL(n, \mathbb{C})$ on a finite dimensional complex vector space V_ρ . Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m . Let $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ . We define the $|\rho, \mathcal{M}$ -slash action of G^J on $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ as follows:

If $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$,

$$(7.2) \quad \begin{aligned} & f|_{\rho, \mathcal{M}}[(M, (\lambda, \mu; \kappa))](\Omega, Z) \\ & := e^{-2\pi i \operatorname{tr}(\mathcal{M}[Z + \lambda\Omega + \mu](C\Omega + D)^{-1}C)} \cdot e^{2\pi i \operatorname{tr}(\mathcal{M}(\lambda\Omega^t\lambda + 2\lambda^tZ + \kappa + \mu^t\lambda))} \\ & \quad \times \rho(C\Omega + D)^{-1}f(M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}), \end{aligned}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta] = {}^t\beta\alpha\beta$ for suitable matrices α and β . We define $\mathbb{D}_{\rho, \mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$(7.3) \quad (Df)|_{\rho, \mathcal{M}}[g] = D(f|_{\rho, \mathcal{M}}[g])$$

for all $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and for all $g \in G^J$. We denote by $\mathcal{Z}_{\rho, \mathcal{M}}$ the center of $\mathbb{D}_{\rho, \mathcal{M}}$.

We define an another notion of Maass-Jacobi forms as follows.

Definition 7.2. A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index \mathcal{M} if it satisfies the following conditions $(MJ1)_{\rho, \mathcal{M}}$, $(MJ2)_{\rho, \mathcal{M}}$ and $(MJ3)_{\rho, \mathcal{M}}$:

- $(MJ1)_{\rho, \mathcal{M}}$ $\phi|_{\rho, \mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.
- $(MJ2)_{\rho, \mathcal{M}}$ f is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho, \mathcal{M}}$ of $\mathbb{D}_{\rho, \mathcal{M}}$.
- $(MJ3)_{\rho, \mathcal{M}}$ f has a growth condition

$$\phi(\Omega, Z) = O\left(e^{a \det Y} \cdot e^{2\pi i \operatorname{tr}(\mathcal{M}[V]Y^{-1})}\right)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

The case $n = 1$, $m = 1$ and $\rho = \det^k (k = 0, 1, 2, \dots)$ was studied by R. Bendt and R. Schmidt [1], A. Pitale [16] and K. Bringmann and O. Richter [3]. The case $n = 1$, $m = \text{arbitrary}$ and $\rho = \det^k (k = 1, 2, \dots)$ was dealt with by C. Conley and M. Raum [5]. In [5] the authors proved that the center $\mathcal{Z}_{\det^k, \mathcal{M}}$ of $\mathbb{D}_{\det^k, \mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k, \mathcal{M}}$, the so-called *Casimir* operator which is a $|\det^k, \mathcal{M}$ -slash invariant differential operator of degree three. Bringmann and Richter [3] considered the Poincaré series $\mathcal{P}_{k, \mathcal{M}}$ (the case $n = m = 1$) that is a *harmonic* Maass-Jacobi form in the sense of Definition 7.2 and investigated its Fourier expansion and its Fourier coefficients. Here the *harmonicity* of $\mathcal{P}_{k, \mathcal{M}}$ means that $\mathcal{C}^{k, \mathcal{M}}\mathcal{P}_{k, \mathcal{M}} = 0$, i.e., $\mathcal{P}_{k, \mathcal{M}}$ is an eigenfunction of $\mathcal{C}^{k, \mathcal{M}}$ with zero eigenvalue. Conley and Raum [5] generalized the results in [16] and [3] to the case $n = 1$ and m is arbitrary.

Remark 7.2. In [2], Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They also introduce an analogue of Kohnen's plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over K .

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